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Nash equilibrium in duopoly with products defined by two characteristics

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This article analyzes the analogue of Hotelling's duopoly model when products are defined by two characteristics. Using the assumptions of the original model of Hotelling, we show that demand and profit functions are continuous for a wide class of utility functions. When the utility function is linear in the Euclidean distance in the space of characteristics, a noncooperative equilibrium in prices exists for all symmetric locations of firms. This is in contrast to the result in the one-characteristic model where a noncooperative equilibrium exists only when products are very different. The noncooperative equilibria are calculated and fully characterized. In contrast with the one-dimensional model of Hotelling, where equilibrium prices were constant irrespective of distance (of symmetric locations), here equilibrium prices tend to zero as the distance between products approaches zero.

1. Introduction

■ Hotelling (1929) pioneered the study of models with differentiated products defined by their characteristics. He examined a duopoly with each firm's producing a distinct product differentiated by a single characteristic. Assuming a specific utility function and a uniform distribution of preferences, Hotelling analyzed the existence of a Nash equilibrium in prices, given fixed varieties. D'Aspremont *et al.* (1979) have proved that, in the original model of Hotelling, no Nash equilibrium in prices exists unless the products offered are relatively far apart. The reason is that for relatively close products it pays each competitor to undercut his opponent and to capture the whole market. This tendency is stronger, the closer are the products offered. Once the best reply involves undercutting, no Nash equilibrium exists.

Several variations of Hotelling's one-characteristic model have arisen that attempt to deal with the problem of existence of equilibrium. Novshek (1980) assumed that once a firm is undercut, it will react by cutting prices—thereby violating the zero-conjectural variation rule. Salop (1979) showed that undercutting is suboptimal when firms are located symmetrically on a circumference, because the undercutting price is equal to marginal cost. Economides (1984) showed that the region of existence of equilibria widens when a finite reservation price is imposed on Hotelling's model. D'Aspremont *et al.* (1979) and Economides (1986) have shown that the region of existence of equilibrium is enlarged when the convexity of the utility function in the space of characteristics increases.

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This article develops a two-dimensional equivalent of Hotelling's duopoly model.[1] Consumers are distributed according to their most preferred variety on a disk, rather than on an interval $[a, b]$. This model is of interest because in a typical market firms compete with products defined by a number of characteristics. Further, the array of negative results of the one-characteristic model prompts us to examine their generality. We show that, for reasonable utility functions, the demand and profit functions of the competing firms are continuous. This is also true in models of higher dimensionality of the space of characteristics. Although this is a marked improvement over the one-dimensional case where the objective functions are discontinuous, they still lack quasi concavity. Despite the lack of quasi concavity, and the implied discontinuity of the best reply functions, I am able to prove the existence of, and can characterize, a noncooperative equilibrium in pure strategies.

2. The model

■ There are two kinds of commodities, a homogeneous good, m, and a set of potential differentiated commodities, $C = \{x, y | x^2 + y^2 \leq 1\}$, the unit disk. Each differentiated commodity is defined by a point in C. Without possible confusion, point x in C will be identified with one unit of commodity x. We can think of each point x in C as representing an amount of the characteristic that one unit of commodity x embodies.

Commodity bundles (z, m) in $C \times I\!R$ have one unit of a differentiated commodity, either one unit of x or one unit of y, and an amount of the homogeneous good m.[2] Consumption agent w is endowed with a utility function $U_w(z, m)$. U_w is separable:

$$U_w(z, m) = m + V_w(z).$$

$V_w(z)$ is single-peaked at $z = w$. We can write it as $V_w(z) = V_w(w) - f(d(z, w))$, where $f(d)$ is linear in d, when $d(z, w)$ is a distance function and $f(0) = 0$. $f(\cdot)$ is sometimes called the "transportation cost function," since it represents the transportation cost in the formally equivalent location model.

Consumers are price-takers. They maximize their utility, given the commodities and prices offered in the market. Given a choice of x at price P_x and y at price P_y, consumer w buys the commodity that minimizes the sum of price and transportation costs. Consumers are distributed uniformly on C, the product space, according to their most preferred commodity.

There are two firms. Firm 1 produces commodity x, and firm 2 produces product y. There are no costs of production, but the results would be unchanged if there were constant marginal costs. As is seen later, increasing marginal costs strengthen the existence arguments.

In the market game the strategic variable of each firm is the price of its product. The products x, y are fixed. Firm 1's (respectively 2's) profit function Π_1 (respectively Π_2) is a function of its own price P_1 (respectively P_2) with parameters the price of the opponent firm and both varieties (locations) x, y:

$$\Pi_1(P_1, P_2; x, y) \equiv P_1 D_1(P_1, P_2; x, y),$$

$$\Pi_2(P_2, P_1; x, y) \equiv P_2 D_2(P_2, P_1; x, y).$$

The solution concept is the noncooperative equilibrium. A pair of prices (P_1^*, P_2^*) is a noncooperative equilibrium if

[1] An early attempt to model duopoly in the two-dimensional space of characteristics was made by Fetter (1924).

[2] This assumption, and others made below, follow exactly Hotelling (1929) and are made in the attempt to examine the exact analogue in two dimensions of the one-dimensional duopoly of Hotelling.

$$\Pi_1(P_1^*; P_2^*, x, y) \geq \Pi_1(P_1; P_2^*, x, y), \qquad \text{all } P_1,$$

$$\Pi_2(P_2^*; P_1^*, x, y) \geq \Pi_2(P_2; P_1^*, x, y), \qquad \text{all } P_2,$$

where P_1 and P_2 lie in an interval $[0, k]$.

3. Continuity of the demand and profit functions

■ In two dimensions there is no natural "distance" in a space of characteristics.[3] The Euclidean distance can be used, but many other possibilities are available. For example, one can use the "block metric"[4] so that the distance between $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|. \tag{1}$$

If the consumers are uniformly distributed over a rectangle parallel to the directions of the coordinate system, and the firms are located on a line parallel to the coordinate system, then the problem with the block metric is equivalent to the one-dimensional problem with [0, 1] as the commodity space.[5]

One important difference between the one- and two-characteristic cases is that here there are metrics such that, if the utility function is linear in distance, and there are no atoms in the distribution of consumers, then the demand and profit functions are continuous. We examine next how such a possibility arises.

For a general distance function $d(a, b)$ the boundary $z = (z_1, z_2)$ between the market served by firms 1 and 2 fulfills

$$P_1 + d(x, z) = P_2 + d(y, z). \tag{2}$$

The demand for firm 2 is the measure of consumers "located" on the same side of (2) with y. The demand for firm 2, D_2, is zero when $P_2 - P_1 > d(x, y)$. Continuity of D_2 at $P_2 = P_1 + d(x, y)$ depends on whether $D_2(P_2)$ tends to zero as P_2 tends to $P_1 + d(x, y)$ from below. This depends on the limit of the boundary defined by (2) as P_2 tends to $P_1 + d(x, y)$ from below. If in the limit the boundary has no "thickness" (i.e., if it coincides with the extension of the straight line xy), then $\lim_{P_2 \to P_1 + d(x,y)^-} D_2 = 0$, and there will be no discontinuity. (See Figures 1(a) and 1(b).) On the other hand, if in the limit the boundary has "thickness" (i.e., if it does not coincide with the extension of the line segment xy), then the limit of D_2, as P_2 tends to $P_1 + d(x, y)$ from below, is not zero, and therefore there is a discontinuity (since $D_2 \to 0$ as $P_2 \to P_1 + d(x, y)^+$). At price $P_2 = P_1 + d(x, y)$ the consumer located at y is taken over by firm 1. Hence, the possible discontinuity occurs when firm 1 takes over the consumer located at the base of firm 2, at point y, and similarly, when firm 2 takes over the consumer located at the base of firm 1, at point x. At price $P_2 = P_1 + d(x, y)$ the boundary between the markets fulfills (from (2))

$$d(x, z) = d(x, y) + d(y, z). \tag{3}$$

For continuity of the demand and profit functions equation (3) must be fulfilled only by points z that are collinear to x and y. Thus, we have the following proposition.

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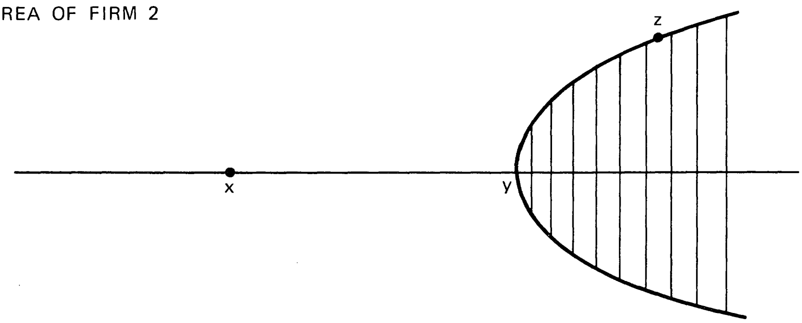
[3] A distance function $d(a, b)$ has to fulfill $d(a, a) = 0$, $d(a, b) = d(b, a) = d(a - b, 0)$, $d(a, b) \geq 0$, and $d(a, b) + d(b, c) \geq d(a, c)$.

[4] The names "block metric" and "taxi metric" are used by urban economists for this distance function, because in a locational framework it signifies that one has to go around the block from x to y.

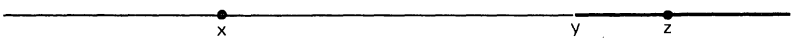
[5] The essential point is that the boundary is orthogonal to xy for any prices. At prices P_1, P_2 let $z = (z_1, z_2)$ be at the boundary. Then $P_1 + |x_1 - z_1| = P_2 + |y_1 - z_1|$, since $|x_2 - z_2| = |y_2 - z_2|$, and therefore $z_1 = (y_1 + x_1 + P_2 - P_1)/2$, which defines a line orthogonal to xy that moves parallel to itself when prices change.

FIGURE 1

MARKET AREA OF FIRM 2



(a) POSITIVE MEASURE EVEN WHEN MARKET BOUNDARY PASSES THROUGH y



(b) ZERO MEASURE WHEN THE MARKET BOUNDARY PASSES THROUGH y

Proposition 1. Let the distance function measuring the loss of utility in the space of characteristics of dimension $n \geq 2$ be such that the triangle inequality holds as equality only for collinear points. Then the demand and profit functions are continuous.

This property holds for an important family of metrics, the l^p metrics, $p \in (1, \infty)$, defined by $d_p(x, y) = [\sum_{i=1}^{n} |x_i - y_i|^p]^{1/p}$, which include the Euclidean metric for $p = 2$.

Proposition 2. The triangle inequality holds as equality only for collinear points when distance is measured by a metric of the l^p family, $p \in (1, \infty)$. The Euclidean metric that is a member of this class for $p = 2$.

Proof. See the Appendix.

Note that this result is not restricted to the two-dimensional space.

We have already seen that the above property does not hold for the "block metric." It seems intuitive, however, that for every metric for which this property does not hold, there exists another neighboring metric for which the property holds. Thus we conjecture that the above property holds generically.[6]

In the light of these results, the observation of discontinuities in the profit function (that arise when the block metric is used in a multidimensional space as well as in its equivalent standard one-dimensional paradigm) is very rare.

The continuity of the profit function guarantees the existence of equilibrium in mixed strategies in prices. The existence of equilibrium in pure strategies is not immediate. The next section examines this question.

4. Noncooperative equilibrium in prices for symmetric locations and Euclidean distance

■ To be able to see the qualitative differences between the results in one and two dimensions, we study a two-dimensional model that is the direct analogue of the original, one-

[6] The property discussed in Propositions 1 and 2 is equivalent to strict convexity of (unit) balls. Our conjecture is that for any distance function that does not create a strictly convex unit ball, there exists a "neighboring" distance function that creates a strictly convex unit ball.

dimensional model of Hotelling (1929). Let consumers be distributed uniformly (according to their most preferred variety) on a unit disk. We take their utility functions to be linear in the Euclidean distance in the product space. We take the reservation price of all consumers to be sufficiently high to ensure that all consumers buy the differentiated product. For simplicity of calculation we shall take the two competitors in the industry to be symmetric with respect to the center of the market, at locations $x = (c, 0)$, $y = (-c, 0)$.

Then consumer w chooses the minimum of $P_1 + r\|(c, 0) - (w_1, w_2)\|$ and $P_2 + r\|(-c, 0) - (w_1, w_2)\|$. The consumers that are indifferent between buying from firm 1 or from firm 2 are located on a hyperbola. This hyperbola is the boundary in space between the locations of all consumers that buy from firm 1 and the locations of all consumers that buy from firm 2. This boundary is the solution (z_1, z_2) of

$$P_1 + r((c - z_1)^2 + z_2^2)^{1/2} = P_2 + r((c + z_1)^2 + z_2^2)^{1/2}, \tag{4}$$

which is equivalent to

$$z_1^2/a^2 - z_2^2/(c^2 - a^2) = 1, \qquad \text{where} \qquad a = |P_2 - P_1|/2r. \tag{5}$$

The market for firm 1, the one that charges the higher price, is shown shaded in Figure 2. This area, bounded between the circle of unit radius and the hyperbola of equation (5), denoted in Figure 2 by $(BEFD) = 2(EBD)$, is equal to

$$T(a) = \arcsin\,[(c^2 - a^2)^{1/2}(1 - a^2)^{1/2}/c] - a(c^2 - a^2)^{1/2}\log_e\{[(1 - a^2)^{1/2} + (1 + c^2 - a^2)^{1/2}]/c\}.$$

Normalizing by setting $r = 1/2$, we can write the demand for firm 1 as

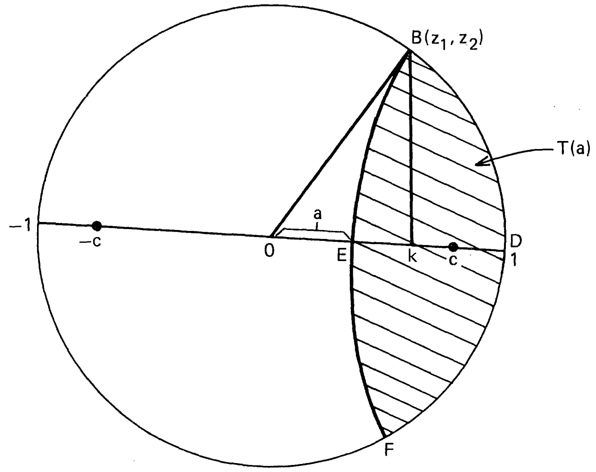
$$\begin{aligned} D_1(P_1, P_2) &= \pi & P_1 &\le P_2 - c \\ &= \pi - T(P_2 - P_1) & P_2 - c \le P_1 &< P_2 \\ &= T(P_1 - P_2) & P_2 \le P_1 &< P_2 + c \\ &= 0 & P_2 + c &\le P_1. \end{aligned}$$

The demand for firm 2 is $D_2(P_1, P_2) \equiv \pi - D_1(P_1, P_2)$.

By Propositions 1 and 2 there are no discontinuities of the demand function, since here we use the Euclidean distance function. At the price $P_2 = P_1 - c$, at which firm 2 wins over the consumers located at the opponents' production point, c, it also takes over all

FIGURE 2

MARKET AREA OF FIRM 1 WHEN IT QUOTES A HIGHER PRICE THAN FIRM 2



consumers located in the interval $(c, 1]$. But this interval has no "thickness" in two dimensions, i.e., it has measure zero. Thus, $T(a)$ is continuous at $a = c$.

The first-order conditions for noncooperative equilibrium,

$$\partial\Pi_1(P_1, P_2)/\partial P_1 = 0, \qquad \partial\Pi_2(P_1, P_2)/\partial P_2 = 0$$

under symmetry are equivalent to:

$$D_1(P^*, P^*) + P^*\partial D_1(P^*, P^*)/\partial P_1 = 0. \tag{6}$$

The solution of this equation is:

$$P^*(c) = -T(0)/T'(0) = \pi/\{2((c^2+1)^{1/2}/c + c\log_e(1+(1+c^2)^{1/2}) - c\log_e c)\}. \tag{7}$$

$p^*(c)$ is strictly increasing in c, with $\lim_{c\to 0} P^*(c)/c = \pi/2$ and $\lim_{c\to\infty} P^*(c) = \pi/4$.

As seen in Figure 3, the profit function is not quasi-concave. The existence of equilibrium is based on the fact that "undercutting," i.e. playing $P_1 = P^u \equiv P^* - c$, is inferior to playing $P_1 = P^*$. This is established by direct computation.[7]

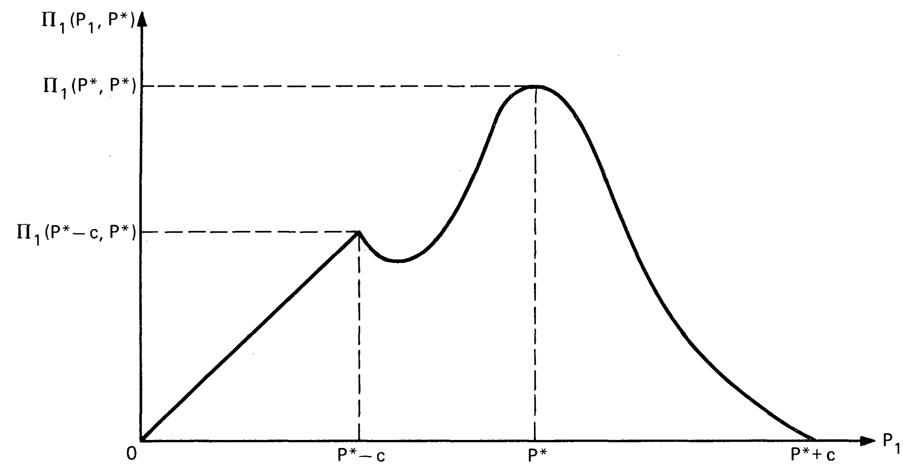
Theorem. For products differentiated in two dimensions, let two firms be located symmetrically on an axis through the center of the market. Let c be the distance of each firm from the center. Let the reservation price of consumers be sufficiently high that all consumers buy the differentiated product. Then a symmetric noncooperative equilibrium exists for any $c > 0$ at prices given by (7).

To understand this result better we compare it with the results of the one-dimensional model of Hotelling (1929) as corrected by d'Aspremont *et al.* (1979) for the case when firms are close to each other (small c). In these models existence of equilibrium can fail for small c.

In the one-dimensional model for symmetric locations the "equilibrium" prices are constant at $P_1^* = P_2^* = r$ and "equilibrium" profits are $\Pi_1^* = \Pi_2^* = r/2$.[8] The undercutting

FIGURE 3

PROFIT FUNCTION OF FIRM 1 FOR SYMMETRICALLY LOCATED FIRMS



[7] The detailed computations of this result and of the results on the properties of $P^*(c)$ are available from the author upon request.

[8] In the present two-dimensional model we have normalized the slope of the "transportation cost function" at $r = ½$. The normalization does not affect the arguments, but should be kept in mind when making comparisons of equilibrium prices across models.

price is $P^u = r(1 - 2c)$, so that $P^*/P^u = 1 - 2c \to 1$ as $c \to 0$. Thus, when locations are close, i.e., for small c, the equilibrium and undercutting prices are nearly identical. Then clearly there is an advantage to undercut, since the demand at the undercutting price is twice that at the "equilibrium" price.

In the present two-dimensional model, the equilibrium prices as c goes to zero are approximated by $\pi c/2$, so that $P^* \to 0$ as $c \to 0$. The undercutting price is approximated by $(\pi/2 - 1)c$, and similarly goes to zero as $c \to 0$. The ratio $P^*/P^u \cong \pi/(\pi - 2) > 2$ for small c. Therefore, a significant percentage price cut is required to undercut the opponent. This relative price cut has to be compared with the doubling of demand that results from undercutting. Here $P^*/P^u > 2 = D^u/D^*$, and this implies that $\Pi^* > \Pi^u$, i.e., that equilibrium profits are larger than undercutting profits.[9]

In evaluating these models as useful abstractions of the working of markets, it is reasonable to require that the equilibrium prices of products of nearly identical specifications be very close to marginal cost. This requirement holds in this model and fails in the one-dimensional analogue of Hotelling (1929). This is necessary, but not sufficient, for the existence of the noncooperative equilibrium. A rapid rate of approach of prices to zero as the distance between the products goes to zero is required, as seen in the discussion of the previous paragraph.

The existence result of the disk market can be used to derive conclusions on the existence of equilibrium in markets of other geometric shapes. Suppose that we modified the disk by the symmetric addition of an area of size β on the hinterland of each firm.[10] We show that equilibrium existence is guaranteed in the new market for $\beta < 0$, i.e., when symmetric areas are subtracted from hinterlands. Equilibrium demand, $D^* \equiv D_1(P^*, P^*)$, increases by $\Delta D^* = \beta$, while total and undercutting demand increases by $\Delta D^u = 2\beta$. Equilibrium price changes come through (6). Differentiating (6), we derive $dP^*/dD^* = -1/\partial D_1(P^*, P^*)/\partial P_1 = -1/T'(0) \equiv \tau > 0$. Thus, $\Delta P^* = \beta\tau$, $\Delta P^u = \Delta(P^* - c) = \beta\tau$, and, therefore, the implied profit changes are

$$\Delta\Pi^* = \beta P^* + \beta\tau D^* + \beta^2\tau,$$

$$\Delta\Pi^u = 2\beta P^u + \beta\tau D^u + 2\beta^2\tau,$$

so that $\Delta(\Pi^* - \Pi^u) = \beta(P^* - \tau D^* - 2P^u) - \beta^2\tau = \beta(-2P^u - \beta\tau)$. Therefore, any small decrease in the market area away from the equilibrium border ($\beta < 0$ and $|\beta| < 2P^u/\tau$) increases the difference between the equilibrium and the undercutting profits. Thus, the existence of equilibrium when the market area is a disk implies the existence of equilibrium for a disk truncated at the tails.[11] The opposite changes, small increases in the market area away from the equilibrium border ($\beta > 0$, small), decrease the difference between equilibrium and undercutting profits. Thus, the expansion of the "tails" of the disk can lead to nonexistence of equilibrium.

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[9] We can also show that the present two-characteristic model with uniform distribution of consumers is equivalent to a one-characteristic model with density function

$$f(x) = -T'(x)/\pi \qquad 0 < |x| < c$$
$$= 0 \qquad c \leq |x| \leq 1.$$

This density for the one-dimensional model is discontinuous, and depends on the locations of the products (c), rather than being exogenous. This illustrates further the differences between the standard one- and two-characteristic models of product differentiation and argues against using the one-characteristic model with uniform distribution as the standard paradigm.

[10] The "hinterland" of firm 1 is traditionally defined as the set of locations $z = (z_1, z_2)$ that fall on the side of firm 1 opposite to that of firm 2, $z_1 \geq c$. For the purposes of the argument in the text, it is important that the truncated hinterland locations be away from the symmetric equilibrium market boundary, located at $z_1 = 0$.

[11] In the above argument we also assume that the proposed truncations do not alter the two-peaked shape of the profit function.

Corollary. Existence of equilibrium for symmetric firm locations is guaranteed for markets of geometric shapes that are derived from the disk through symmetric truncation of small areas in the hinterlands of the firms.[12]

It has become clear that the question of existence of equilibrium in locational duopolies is essentially a question of the balance among the weight of consumers at the edges of the market, the locational rate of change of the boundary, and the shape of the market area at the equilibrium boundary. Another dimension can be added to these models by allowing for some elasticity of demand. Such a change would not significantly change the arguments made so far. And it is important to understand that there can be diverse characterizations of locational duopolies, even when the elasticity of the industry demand is zero.

Compared with the one-dimensional model, in two dimensions the availability of many admissible distance functions allows for a variety of rates of change of the boundary between the market areas, and thus there can be a variety of market structures for the same strategic form. Such variety is further enhanced by the availability of different geometric market areas, the shape of which near the equilibrium boundary plays a key role in the determination of equilibrium. Therefore, it is not so surprising that the result here is different from that in the one-dimensional model where these possibilities did not exist.

5. Concluding remarks

■ We analyzed the analogue of Hotelling's duopoly model when products are defined by two of their characteristics. Contrary to the results of the one-characteristic model, in the two-characteristic model the demand and profit functions are continuous everywhere for a reasonable class of distance functions in the space of characteristics that includes the Euclidean distance. Further, in this model a Nash equilibrium in prices exists for all symmetric varieties. This is in contrast to the results of d'Aspremont *et al.* (1979), where a Nash equilibrium in prices does not exist for relatively close locations in the one-characteristic model. Computational limitations do not allow us to analyze the case of nonsymmetric varieties in this model. The existence result of the symmetric case leads us to speculate, however, that a Nash equilibrium in prices exists for nonsymmetric varieties that are close to symmetric ones. The question of the existence of subgame perfect equilibria in a two-stage location-price game with products defined by two characteristics remains open.

Appendix

■ The proof of Proposition 2 follows.

Proof of Proposition 2. This proof is for a two-dimensional space of characteristics, but is essentially the same for space of higher dimension. We show that for l^p metrics

$$\|x-z\| = \|x-y\| + \|y-z\| \tag{A1}$$

is true if and only if x, y, and z are collinear, i.e., whenever

$$(z_2-y_2)(y_1-x_1) = (y_2-x_2)(z_1-y_1). \tag{A2}$$

That A2 implies A1 is trivial. We next prove that A2 implies A1. Let x, y, and z lie on a straight line, and let z^* lie on the perpendicular to the line at z, so that z is the projection of z^* on the line xyz. Let the first of the coordinate axes coincide with the line xyz, so that $|y_1 - x_1| = m$, $|y_2 - x_2| = 0$, $|z_1 - y_1| = 1$, $|z_2 - y_2| = 0$, $|z_1^* - z_1| = 0$, $|z_2^* - z_2| = \theta$. It is sufficient to prove that $\theta > 0 \rightarrow \|x - z^*\| < \|x - y\| + \|y - z^*\|$. Now $\|x - z^*\| < \|x - y\| + \|y - z^*\| \Leftrightarrow \{(m + w)^p + \theta^p\}^{1/p} < m + \{w^p + \theta^p\}^{1/p}$. Let

$$f(\theta) = m + (w^p + \theta^p)^{1/p} - \{(m + w)^p + \theta^p\}^{1/p}.$$

[12] There are no implications about markets that differ in shape from the disk in the neighborhood of the symmetric equilibrium (around the vertical axis through the center), because they differ in $\partial D_1(P^*, P^*)/\partial P_1$.

Clearly, $f(0) = 0$. It is sufficient to prove that $\theta > 0$ implies that $f'(\theta) > 0$. Since

$$f'(\theta) = p\theta^{p-1}[(w^p + \theta^p)^{(-1+1/p)} - \{(m+w)^p + \theta^p\}^{(-1+1/p)}], \qquad f'(\theta) > 0 \Leftrightarrow (w^p + \theta^p)^{p-1} < ((m+w)^p + \theta^p)^{p-1},$$

which is true for all p such that $\infty > p > 1$.

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