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Segmenting the Market: The Monopolist's Optimal Product Mix*

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In an industry producing products which differ in quality, to consumers who vary in their willingness to pay, it may happen that only a bounded number of producers can coexist at (noncooperative price) equilibrium; in other words, the industry is a natural oligopoly. We are here concerned with a special example, in which only one producer can survive. Our focus of interest in the present paper is to examine this monopolist's optimal product range. Depending on the dispersion of consumers' willingness to pay (income), either (i) the monopolist will find it optimal to segment the market completely, offering the maximum number of products permitted or (ii) the monopolist will offer only a single product. The precise nature of this switch of policy, which occurs at a certain critical distribution of consumer incomes (willingness to pay), is explored fully. *Journal of Economic Literature* Classification Numbers: 022, 600. © 1986 Academic Press, Inc.

Consider a group of consumers who choose between alternative products which differ in quality. Suppose these consumers vary in their willingness to pay for a higher quality product, whether because of income differences

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(richer consumers being willing to pay more) or because of differences in tastes. Assume that each of these products may be produced at some constant level of unit variable cost (marginal cost) which may in general vary with product quality. In a sequence of papers modelling this situation [2, 3, 5, 6], we have pointed out the following basic property. *If unit variable cost rises only "slowly" with improvements in quality—relative to consumers' willingness to pay for quality improvements—then an upper bound exists to the number of firms which enjoy prices above unit variable cost, at a noncooperative price equilibrium.* Thus, given any fixed cost (of "entry" say), however small, the number of firms which can survive is bounded; this upper bound depends in general on the distribution of consumers' "willingness to pay," and on the precise relationship between unit variable cost, and quality. In the present paper, we will be concerned with the most extreme case, in which the dispersion of consumers' willingness to pay, combined with the relationship between variable costs and quality, imply that only a single firm will survive in the market. In other words, we here consider the case of a "natural monopoly."

The condition we adduced on unit variable costs above is certainly satisfied if such costs are everywhere zero; we here assume this to be the case, for simplicity. Given this, the number of firms which can survive depends only on the dispersion of consumers' willingness to pay. Taking this to be simply a function of income, it turns out that the "narrower" the range of incomes, the fewer the number of competing firms which can coexist. Thus we begin our analysis by determining exactly how "narrow" the spread of incomes must be in order to guarantee that only a single firm can survive. If this condition is satisfied, we have a natural monopoly, and no ad hoc restrictions on entry are further required to justify our focusing on a single producer.

In earlier papers, we confined our analysis to single-product firms; in the present paper we turn to the case of multi-product firms. Our main focus of interest however is confined to the detailed study of the range of products (qualities) produced by a monopolist. Our central theme is that the optimal strategy again depends crucially on the range of incomes in the economy. We have already required it to be sufficiently narrow to admit only one producer; suppose now it is narrowed even further. Suppose the monopolist is permitted to produce at most n (possibly infinite) products. Then, at first, when the range is still sufficiently "broad," it is optimal for the monopolist to segment the market by selling n products. However, as the range of incomes becomes narrower, there occurs a critical point beyond which the firm's optimal strategy is to sell the same product to all consumers—i.e., not to segment the market at all. To see the precise mechanism through which this "catastrophe" occurs, it is necessary to explore in some detail the associated "discrete" problem, in which the monopolist is constrained to offer n products, and to examine how the solution converges to the optimal policy for the "continuum" case, as n becomes arbitrarily large.

Rosen [4]. There, it is shown that the optimal strategy will involve producing a range of qualities and setting prices according to a price–quality schedule which is steeper than the unit variable cost–quality schedule. As in the present paper, for the case where the optimal strategy is to offer a continuum of products, the monopolist "separates markets, and assigns different customer types to different varieties of goods, thereby permitting partial discrimination among consumers of varying intensities of demands" (see [4, p. 301]).[1] However, they also note that "demand conditions may be such that it does not pay the seller to separate all markets completely, but rather to bunch customers of different tastes onto the same product. This maneuver is accomplished by imparting corners in the price–quality schedule, so that customers with somewhat different marginal valuations of quality find it in their interest to purchase the same quality item" (see [4, p. 301]).

The cost and utility functions employed in the present paper are such that the kind of "bunching" which may arise in the Mussa–Rosen model does not occur. The bunching which occurs in the present paper, is, rather, a consequence of our basic assumption that the unit variable cost schedule is "flat," the same feature which leads to the "natural monopoly" characterization.

The paper is organized as follows. In Section 1, the "finiteness property" of the equilibrium is established for the case of multi-product firms (this is a generalization of the property explored in the above-mentioned papers); and the condition for a single surviving firm is derived. We then focus on the optimal policies of that firm when it is restricted to produce at most a given number of products. This is modelled as a two-stage process. First, the firm chooses an optimal price schedule corresponding to a vector of given qualities (Sect. 2). In the second stage it chooses the optimal finite vector of qualities by reference to the corresponding optimal price schedule (Sect. 3). In Section 4, the study of the monopolist's policy is extended to cope with the case where the firm is allowed to produce a continuous product line. Furthermore, the corresponding policy is shown to be the limit of the n-product optimal policy when n becomes large. Section 5 provides a summary of our results and some conclusions.

[1] The discrete case has been explored recently by Itoh [1].

1. A Natural Monopoly

Consider a continuum of consumers identical in tastes but differing in income; incomes are uniformly distributed on some support, so that the density function equals unity on $[a, b]$, where $a > 0$, and zero otherwise. The assumption that a be strictly positive plays a crucial role, as will become clear presently.

Consumers make indivisible and mutually exclusive purchases from among a number of substitute goods, in the sense that any consumer either makes no purchase, or else buys exactly one unit of one of the goods. Goods differ in quality, and we denote the utility derived by consuming one unit of product k, and t units of "income"—interpreted as a Hicksian composite commodity, say—as $U(t, k)$; while $U(t, 0)$ denotes the utility derived from consuming t units of income only.

Let the utility function take the form

$$U(t, k) = u_k \cdot t$$

and (1)

$$U(t, 0) = u_0 \cdot t$$

with $u_0 < u_1 < \cdots < u_n$ (the n products are labelled in increasing order of quality). Let

$$r_{k-1,k} = \frac{u_k}{u_k - u_{k-1}}$$

(where $r_{k-1,k} > 1$). Then we may define the income level t_k such that a consumer with this income will be indifferent between good k at price p_k and good $k-1$ at price p_{k-1}. Using (1),

$$t_k = p_{k-1}(1 - r_{k-1,k}) + p_k r_{k-1,k}$$

and (2)

$$t_1 = p_1 r_{1,0}.$$

The utility function defined in (1) has the property that the willingness to pay increases with income, so that consumers of income above t_k prefer to buy some higher quality good, and conversely. Thus we may partition the consumers by income, into intervals corresponding to the market areas of successive products, higher income bands being associated with higher quality products.

Consider now the supply side of the model. For each firm in the industry, we may identify the set of products which are offered by that firm; among them, consider any "band" of adjacent products, i.e., any sequence of products of different qualities such that no product in between is produced by another firm; we label by k, $k+1, \ldots, l$, with $k \leqslant l$, the products inside any such band. A firm may of course produce several such bands, i.e., the product ranges of various firms might overlap. But we focus here on one such band (what we wish to take into account is the fact that some rival firms may produce the next lowest quality $k-1$, and the next highest $l+1$).

Suppose, for the moment, that more than one firm survives with a positive market share at a Nash equilibrium in prices. Then, at least one of these firms does not sell the lowest quality product on the market (for, otherwise, the equilibrium price of that product would be equal to zero as in Bertrand). Consider any band of adjacent products k, $k+1, \ldots, l$ which that firm produces. Its revenue takes the form, if l is not the top quality:

$$p_k(t_{k+1}-t_k)+p_{k+1}(t_{k+2}-t_{k+1})+\cdots+p_l(t_{l+1}-t_l)+\tilde{R},$$

while, if l is the top quality:

$$p_k(t_{k+1}-t_k)+p_{k+1}(t_{k+2}-t_{k+1})+\cdots+p_l(b-t_l)+\tilde{R},$$

where $\tilde{R}$ (independent of $p_k, \ldots, p_l$) is the revenue the firm derives from other products.

Differentiating w.r.t. $p_k, \ldots, p_l$ we obtain the first-order conditions for revenue maximization; writing the market share of product h as $M^h = t_{h+1} - t_h$, these take the form, if l is not the top quality:

$$\begin{aligned}
&M^k - p_{k+1}(1-r_{k,k+1}) + p_k(1-r_{k,k+1}) - p_k r_{k-1,k} = 0\\
&M^{k+1} - p_{k+2}(1-r_{k+1,k+2}) + p_{k+1}(1-r_{k+1,k+2})\\
&\qquad - p_{k+1}r_{k,k+1} + p_k r_{k,k+1} = 0\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\
&M^{l-1} - p_l(1-r_{l-1,l}) + p_{l-1}(1-r_{l-1,l})\\
&\qquad - p_{l-1}r_{l-2,l-1} + p_{l-2}r_{l-2,l-1} = 0\\
&M^l - p_l r_{l-1,l} + p_{l-1}r_{l-1,l} + p_l(1-r_{l,l+1}) = 0,
\end{aligned}$$

while, if l is the top quality, the last equation is replaced by

$$M^l - p_l r_{l-1,l} + p_{l-1} r_{l-1,l} = 0.$$

Adding these equations, we obtain, if l is not the top quality:

$$M^k + \cdots + M^l = p_k r_{k-1,k} - p_k + p_l r_{l,l+1},$$

while, if l is the top quality:

$$M^k + \cdots + M^l = p_k r_{k-1,k} - p_k + p_l.$$

Thus, in either case, since $p_l r_{l,l-1} \geqslant p_l \geqslant p_k$,

$$M^k + \cdots + M^l \geqslant p_k r_{k-1,k} \geqslant t_k \geqslant a.$$

Hence, at the Nash equilibrium, this band of products enjoys a market share of at least size $a > 0$, and so, it follows that each firm on the market, except possibly the one with the lowest quality band, enjoys a market share which is bounded from below by a. Thus, *there exists a bound, depending on the distribution of income, to the number of firms which can coexist with positive market shares and prices exceeding variable cost* (*i.e., positive prices*) *at a Nash equilibrium in prices.* In particular, when $b < 2a$, only one firm can "survive" (in the sense adduced in the introduction): we have a natural monopoly. Moreover, this ensures that the market is "covered" in the sense that all consumers buy one of the products offered.

This outcome is brought about via the following mechanism: price competition between firms drives the price of one of the "top" firm's products down to a level such that even the poorest consumer prefers to buy that product at a positive price rather than purchase any of the excluded products, even at price zero. It is the case, therefore, that the presence of an "excluded" rival firm has an effect on the configuration of market prices. However, since this rival firm will earn zero revenue, we here simply assume that no such firm enters. (This question of entry is formulated rigorously by Shaked and Sutton in [5].)

The above demonstration of the "finiteness property" uses the fact that the lower bound to income, a, is strictly positive. If this is not so, then, in our present "zero cost" setting, we could insert an unbounded number of firms, each with a positive market share at equilibrium. (Labelling firms in decreasing order of quality, we would find that successive firms sold to increasingly narrower bands of "poorer" consumers). Once we move beyond the "zero cost" framework, however, the condition $a > 0$ is replaced by a more general condition on the unit variable cost schedule, and the requirement that $a > 0$ emerges as a special, limiting, case of this more general condition (for details, the reader is referred to Shaked and Sutton [6].

2. The Monopolist's Pricing Policy

We will henceforward assume that $b<2a$ so that only one firm is present. Suppose this monopolist sells n products of quality $u_1, u_2,..., u_n$ in $[u_0, \bar{u}]$. He then chooses $p_1,..., p_n$ to maximize his revenue

$$R = p_1(t_2 - a) + p_2(t_3 - t_2) + \cdots + p_n(b - t_n)$$

on $\mathscr{D} = \{(p_1,..., p_n)\colon p_k \geqslant 0 \text{ for } k = 1,..., n,\ t_2 \geqslant a,\ t_{k+1} \geqslant t_k \text{ for } k = 2,..., n-1 \text{ and } t_n \leqslant b\}$.

We know the poorest consumer buys product 1—the market is "covered." Therefore, it is always optimal for the firm to set a price for product 1 at a level such that the poorest consumer is indifferent between buying product 1 and making no purchase (for, otherwise, raising p_1 raises revenue since all consumers who hitherto bought a higher quality product, at a higher price, continue to do so, while previous consumers of good 1 pay more, whether they continue to buy 1 or switch to good 2). Thus

$$\bar{p}_1 = \frac{a}{r_{0,1}}. \tag{3}$$

In what follows, we determine the optimal values of $p_2,..., p_n$ neglecting the constraint $(p_1,..., p_n) \in \mathscr{D}$.

Differentiating R with respect to p_k, for $k = 2,..., n-1$, and using (2), we obtain

$$2M^k = \bar{p}_{k+1} - \bar{p}_{k-1}. \tag{4}$$

Define $s_{k-1,k} = 2r_{k-1,k} - 1 = (u_k + u_{k-1})/(u_k - u_{k-1})$. Then (4) becomes

$$(s_{k-1,k} + s_{k,k+1})\,\bar{p}_k = s_{k-1,k} \cdot \bar{p}_{k-1} + s_{k,k+1} \cdot \bar{p}_{k+1}, \tag{4'}$$

so that p_k is represented as a weighted average of $\bar{p}_{k-1}$ and $\bar{p}_{k+1}$. Similarly for $k = n$, we obtain

$$2M^n = b - \bar{p}_{n-1} \tag{5}$$

or

$$(s_{n-1,n} + 1)\,\bar{p}_n = s_{n-1,n}\,\bar{p}_{n-1} + b. \tag{5'}$$

The function R is strictly concave in $\mathbb{R}^{n-1}$ as R is quadratic and $R \to -\infty$ for $\|(p_2,..., p_n)\| \to \infty$, where $\|\cdot\|$ denotes the Euclidean norm. Consequently, the solution to the system (3), (4'), and (5') maximizes R.

Solving the system of equations (4′) by induction yields

$$\bar{p}_k = \bar{p}_1 + s_{1,2}\left[\frac{1}{s_{1,2}} + \frac{1}{s_{2,3}} + \cdots + \frac{1}{s_{k-1,k}}\right](\bar{p}_2 - \bar{p}_1). \tag{6}$$

As $\bar{p}_1$ is given by (3), the only remaining unknown in (6) is $\bar{p}_2$. Replacing p_{n-1} and p_n by (6) in (5′) we obtain the optimal value of $\bar{p}_2$, i.e.,

$$\bar{p}_2 = \bar{p}_1 + \frac{b - \bar{p}_1}{s_{1,2}B} \tag{7}$$

where

$$B = 1 + \frac{1}{s_{1,2}} + \frac{1}{s_{2,3}} + \cdots + \frac{1}{s_{n-1,n}} = 1 + \sum_{i=2}^{n}\left\{\frac{u_i - u_{i-1}}{u_i + u_{i-1}}\right\}.$$

For $k = n$, we have

$$\bar{p}_n = b - \frac{b - \bar{p}_1}{B}. \tag{8}$$

This determines the vector $(\bar{p}_2,..., \bar{p}_n)$ which maximizes the function R in $\mathbb{R}^{n-1}$ for p_1 given by (3). Let $\bar{t}_k = \bar{p}_{k-1}(1 - r_{k-1,k}) + \bar{p}_k r_{k-1,k}$. It is easily verified that $\bar{t}_k < \bar{t}_{k+1}$ for $k = 2,..., n-1$ and $t_n \leqslant b$. Therefore, for $(\bar{p}_1,..., \bar{p}_n)$ to belong to domain $\mathscr{D}$ we only need $\bar{t}_2 \geqslant a$. This condition will be discussed in the next section.

Finally, the value of R at $(\bar{p}_1,..., \bar{p}_n)$, denoted $\bar{R}$, is determined as follows. Adding Eqs. (4) and (5) yields

$$2(M^2 + \cdots + M^n) = b + \bar{p}_n - \bar{p}_2 - \bar{p}_1.$$

But, since $M^1 + M^2 + \cdots + M^n = b - a$, we get

$$2M^1 = b - 2a - \bar{p}_n + \bar{p}_2 + \bar{p}_1. \tag{9}$$

Accordingly, $\bar{R}$ is determined by multiplying each equation of (4), (5), and (9) by the corresponding price $\bar{p}_k$ and by adding them to obtain

$$2\bar{R} = (b - \bar{p}_1)\bar{p}_n + \bar{p}_1(b - 2a + \bar{p}_1). \tag{10}$$

Thus we see that $\bar{R}$ is a function of $u_1, u_2,..., u_n$ through $\bar{p}_1$, $\bar{p}_2$, and $\bar{p}_n$ as given by (3), (7), and (8).

3. The Monopolist's Product Range: I. The Finite Case

We now examine the monopolist's choice of quality, given that he can sell at most n products from a feasible range $[u_0, \bar{u}]$ of qualities. We do it by reference to the revenue functions evaluated at $(\bar{p}_1,..., \bar{p}_n)$, i.e., to $\bar{R}$.

In view of (10), when $\bar{p}_1$ is fixed, $\bar{R}$ is an increasing function of $\bar{p}_n$ only. Thus, it is convenient to carry out the optimization w.r.t. $u_1, u_2,..., u_n$ in two stages: we first consider a constrained optimization problem in $u_2,..., u_n$ subject to the requirement that $u_1 < \bar{u}$ be fixed at some given level. We then proceed to deal with the optimal choice of u_1.

First we notice that it is always optimal for the monopolist to choose $u_n = \bar{u}$ (for, otherwise, raising the quality of product n from u_n to $\bar{u}$, leaving its price, and all other prices and qualities, fixed, the firm increases its revenue: some consumers who hitherto purchased some good $k < n$ at price $p_k < p_n$ now buy $\bar{u}$ at price p_n). Thus, we are left with the following problem: taking as given the number n of products, the quality u_1 of the first product and the quality $\bar{u}$ of product n, what is the optimal vector of intermediate qualities $u_2,..., u_{n-1}$?

Second, we observe that in (10), using (8), $\bar{R}$ depends on $u_2,..., u_{n-1}$ through B and that $\bar{R}$ is an increasing function of B. Hence, given u_1, the optimal vector $(\bar{u}_2,..., \bar{u}_{n-1})$ is that which maximizes B. As $B = 1 + \sum_{i=2}^{n} (u_i - u_{i-1})/(u_i + u_{i-1})$, the first-order conditions yield (the second-order conditions are satisfied)

$$\frac{\sqrt{u_{k-1}}}{u_{k-1} + u_k} = \frac{\sqrt{u_{k+1}}}{u_k + u_{k+1}}, \qquad \text{for} \quad k = 2,..., n-1$$

or (11)

$$u_k = \sqrt{u_{k-1} \cdot u_{k+1}}.$$

In other words, given $u_1 < u_n = \bar{u}$, the optimal quality corresponding to any intermediate product is the geometric mean of the (optimal) qualities of the neighboring products. From that, it follows immediately that *introducing an intermediate quality does in fact raise revenue* (for the domain $u_{k-1} \leqslant u_k \leqslant u_{k+1}$ permits us to set $\bar{u}_k = u_{k-1}$ or $\bar{u}_k = u_{k+1}$ which is equivalent to not introducing product k). Consequently, the monopolist constrained to produce at most n goods, with the lowest fixed at $u_1 < \bar{u}$, will choose to produce n (in the case, that is, where the constraint that we be in $\mathscr{D}$ is disregarded). Moreover, as the system of first-order conditions (11) has a single solution, there is a unique vector $(\bar{u}_2,..., \bar{u}_{n-1})$ which maximizes $\bar{R}$ for any given $u_1 < \bar{u}$; it is expressed as

$$\bar{u}_k = u_1 \cdot q^{k-1}, \qquad k = 2,..., n-1 \tag{12}$$

where $q = (\bar{u}/u_1)^{1/(n-1)}$. It is then easy to check that for $\bar{u}_k$ given by (12) we have

$$s_{k-1,k} = \frac{q+1}{q-1} \tag{13}$$

and

$$B = 1 + \frac{(n-1)(q-1)}{(q+1)}. \tag{14}$$

Going back to (6), it follows that the prices $\bar{p}_k$ corresponding to qualities $\bar{u}_k$ form an arithmetic progression, viz.,

$$\bar{p}_k = \bar{p}_1 + (k-1)(\bar{p}_2 - \bar{p}_1), \qquad k = 3,..., n. \tag{15}$$

For any given value of n, the vector of price and quality variables $(u_2,..., u_{n-1}, p_2,..., p_n)$ which maximizes R, expressed by (7), (12), and (15), has been derived for a given u_1. Thus, the following problem still remains: (i) Does there exist in the range of qualities $[u_0, \bar{u}]$ some values of u_1 such that the corresponding price vector belongs to $\mathscr{D}$? (ii) If yes, what is the optimal value of u_1? (iii) If no, what is the optimal policy for the monopolist?

(i) To answer the first question, we know that it suffices to have $t_2 \geqslant a$. Using (3), (7), and (12), this can be rewritten as

$$\frac{b-a}{a} \geqslant \frac{u_0}{u_1}\left[B\left(1+\frac{1}{q}\right) - 1\right]. \tag{16}$$

The RHS of (16) takes a value greater than one for $u_1 = u_0$, is decreasing in u_1 on the interval $[u_0, \bar{u}]$ and takes a value equal to $u_0/\bar{u}$ at $u_1 = \bar{u}$. Thus, remembering that $(b-a)/a < 1$ by assumption (Section 1), two cases may arise according as the range of income, i.e., $[a, b]$, is "broad," or "narrow," compared to the range of qualities, i.e., $[u_0, \bar{u}]$. In the first case

$$\frac{b-a}{a} > \frac{u_0}{\bar{u}} \tag{*}$$

and (16) holds for some feasible quality in $[u_0, \bar{u})$. In the second case,

$$\frac{b-a}{a} \leqslant \frac{u_0}{\bar{u}} \tag{**}$$

and (16) is never satisfied for any quality in $[u_0, \bar{u})$.

(ii) Assume that condition (*) holds. Let $u_1^{\min} = \min\{u_1 \in [u_0, \bar{u}]: \bar{t}_2(u_1) \geqslant a\}$, where $\bar{t}_2(u_1)$ denotes the optimal t_2 corresponding to a particular value of u_1. As $(b-a)/a < 1$, it must be that $u_1^{\min} > u_0$. Hence $u_1^{\min}$ is a solution to $(b-a)/a = (u_0/u_1)[B(1+(1/q))-1]$. Clearly, $u_1^{\min} < \bar{u}$ and $\bar{t}_2\ (u_1) \geqslant a$ for any $u_1 \in [u^{\min}, \bar{u})$.

It remains to determine the optimal value of u_1, i.e., $\bar{u}_1$, and to check that $\bar{u}_1$ belongs to $[u_1^{\min}, \bar{u})$. For that purpose, we first rewrite $\bar{R}$ as a function of u_1 in (10) by using (3), (7), and (12); then, differentiating w.r.t. u_1, and equating the resulting expression to zero, we obtain

$$-\frac{q}{(q+1)^2}\left(\frac{b-a}{a}\right)^2 + \frac{u_0}{u_1}\left[B - \frac{2q}{(q+1)^2}\right]\frac{b-a}{a} + \left(\frac{u_0}{u_1}\right)^2\left[B - \frac{q}{(q+1)^2} - B^2\right] = 0.$$

Solving this equation directly for u_1 is rather awkward (recall that q is a function of u_1). We proceed indirectly as follows. Setting $(b-a)/a = d$, we get a second-order equation in d, the roots of which are

$$d_1(u_1) = \frac{u_0}{u_1}\left[B\left(1+\frac{1}{q}\right)-1\right], \tag{17a}$$

$$d_2(u_1) = \frac{u_0}{u_1}\,[B(1+q)-1]. \tag{17b}$$

The functions $d_1(u_1)$ and $d_2(u_1)$ are both decreasing on $[u_0, \bar{u}]$ and coincide at $u_1 = \bar{u}$, where $d_1 = d_2 = u_0/\bar{u}$; furthermore $d_1(u_1) \leqslant d_2(u_1)$. Accordingly, for each value of $d > u_0/\bar{u}$, there are two values of u_1, u_1' and u_1'' say, which solve $d\bar{R}/du_1 = 0$: u_1' is obtained from (17a) and u_1'' from (17b). Clearly, $u_1' < u_1''$ and $u_1' \leqslant u_1^{\min}$. It can be shown that u_1'' (u_1') corresponds to a maximum (minimum) of $\bar{R}$. Hence, if $u_1'' \geqslant u_1^{\min}$, $\bar{u}_1$ is the solution u_1'' of

$$\frac{b-a}{a} = \frac{u_0}{u_1}\,[(n-1)(q-1)+q] \tag{18}$$

obtained from (17b) by replacing B by its value; otherwise, for $u_1'' < u_1^{\min}$, we have $\bar{u}_1 = u_1^{\min}$. *When condition (*) is satisfied, the optimal policy of the monopolist with at most n products is to sell n products (whose qualities are given by* (18), (11), *and* $u_n = \bar{u}$ *at prices given by* (3), (6), *and* (7) *evaluated at* $(\bar{u}_1,\ldots,\bar{u})$.)

(iii) Assume now that condition (**) holds. Accordingly, for any u_1 in $[u_0, \bar{u})$, the price vector defined by (3), (6), and (7) evaluated at $(u_1, \bar{u}_2,\ldots,\bar{u})$ does not belong to $\mathscr{D}$ and, therefore, does not correspond to the optimal policy of the firm. However, for any given u_1 in $[u_0, \bar{u}]$, there exists a unique point $(u_2,..., u_{n-1}, p_2,..., p_n)$ which maximizes R. This implies that any optimal policy of the monopolist, corresponding to any u_1 fixed in $[u_0, \bar{u}]$, is such that the price vector belongs to the boundary of $\mathscr{D}$. Thus, for this policy, there must exist $k \in \{1,..., n-1\}$ such that $t_k = t_{k+1}$ (with $t_1 = a$). The optimal revenue of the firm with n products is therefore not affected if product k is dropped from the market. Consequently, the optimal revenue with n products is equal to the optimal revenue with $n-1$ products. As condition (**) is independent of u_1, the optimal policy with $n-1$ is such that the corresponding price vector also belongs to the boundary of some domain $\mathscr{D}$ (which depends on the lowest quality on the market after k has been dropped). By induction, the optimal revenue with n products (with $n-1$, $n-2$,..., 2 products) is equal to the optimal revenue with a single product, and this holds for any $n > 1$. *If condition (**) is satisfied, it is therefore optimal for the monopolist to sell only the highest quality product, i.e.,* $\bar{u}$, *at price* $((\bar{u} - u_0)/\bar{u})a$.

Our central conclusion may be stated briefly: *if* $(b-a)/a > u_0/\bar{u}$, *so that the range of income is broad compared to the feasible quality range, then the firm segments the market by selling the maximum possible number of products. On the other hand, if* $(b-a)/a \leqslant u_0/\bar{u}$, *so that the range of income is narrow compared to the feasible quality range, then the firm bunches all consumers on the top quality product.*

This implies, it seems, a fundamental discontinuity in the number of products as a function of the range of income in the economy. However, this discontinuity is resolved once we examine in detail how the market shares of the various products behave when the range of incomes, or the feasible quality range, shrinks.

Introducing (15) in (4) yields, for $k = 2,..., n-1$,

$$M^k = \bar{p}_2 - \bar{p}_1$$

so that the market shares of all intermediate products are equal. We solve for $\bar{p}_2 - \bar{p}_1$ from (7) and (13), and obtain

$$M^k = \frac{(b - \bar{p}_1) \cdot (q-1)}{(q+1) \cdot B}. \tag{19}$$

If the parameters are such that (*) holds, the firm produces a full range of n products. Now let $\bar{u}$ fall to the value for which (**) is satisfied as an equality. Then $\bar{t}_2 \to a$ and the market share of product 1 vanishes. Furthermore, $\bar{u}_1 \to \bar{u}$ so that $q \to 1$ and the market shares of the intermediate products 2,..., $n-1$ also vanish by (19). Finally, we show that the market share of the top quality expands to cover the entire market, i.e., $\bar{t}_n \to a$. From (2), (13), and (15) it follows that

$$\bar{t}_n = \bar{p}_1 + (n-2)(\bar{p}_2 - \bar{p}_1) + \frac{q}{q-1} \cdot (\bar{p}_2 - p_1). \tag{20}$$

But $\bar{p}_2 - \bar{p}_1 = (b - \bar{p}_1) \cdot (q-1)/(q+1) \cdot B$ and, by (3) and (17b), $b - \bar{p}_1 = b - a + au_0/\bar{u}_1$, whence $\bar{p}_2 - \bar{p}_1 = (au_0/\bar{u}_1)(q-1)$. After substitution in (20) and some manipulations, we get

$$\bar{t}_n = a + \frac{au_0}{\bar{u}_1}(q-1) \cdot (n-1).$$

Since the last term vanishes in the limit, we have $\bar{t}_n \to a$ as required.

Accordingly, there is an underlying continuity in the behavior of the market shares of all products as one moves from (*) to (**). *So long as the range of incomes exceeds a certain value, the firm continues to offer the maximum feasible number of products. But as the range of incomes narrows, the pricing policy of the firm is such that the market share represented by that band of "rich" consumers who buy the top quality product expands continuously; and, at that point where the income range coincides with its critical value, this "top" band coincides with the entire market.*

4. The Monopolist's Product Range. II. The Infinite Case[2]

Since revenue is increasing in the number n of products, it follows that, in the absence of cost, the monopolist will choose an infinite number of products (assuming that condition (*) holds). We here investigate, therefore, the policy to which the monopolist converges in the limit $n \to \infty$.

Since we have fully described the optimal quality vector and pricing scheme for any n, as functions of $\bar{u}_1(n)$, we have now to examine the limiting value of $\bar{u}_1(n)$ as $n \to \infty$. For any fixed u_1, we note that $\lim_{n\to\infty} q = \lim_{n\to\infty}(\bar{u}/u_1)^{1/(n-1)} = 1$. Moreover, with $y = 1/n - 1$,

$$\lim_{n\to\infty} (n-1)(q-1) = \lim_{y\to 0} \left[\left(\frac{\bar{u}}{u_1}\right)^y - 1\right] \Big/ y = \ln\left(\frac{\bar{u}}{u_1}\right).$$

Hence, given (18), the equation defining $u_1 = \lim_{n\to\infty} \bar{u}_1(n)$ can be written as

$$\frac{b-a}{a} = \frac{u_0}{u_1}\left[\ln\left(\frac{\bar{u}}{u_1}\right) + 1\right]. \tag{21}$$

[2] A similar treatment of this problem for the complementary (Mussa–Rosen) case of the model is given by Champsaur and Rochet [7].

Denote by s the value of u_1 which solves this equation. Finally, we show that $\bar{u}_1(n)$ decreases with n and so converges monotonically to s. To see this, it suffices to examine the behavior of $(n-1)(q-1)+q$ in (18), holding u_1 constant. Let $\bar{u}/u_1 = \beta \geqslant 1$ and $x = n-1$. Then $(n-1)(q-1)+q$ can be written as $f(x) = (x+1) \cdot \beta^{1/x} - x$. Differentiating $f(x)$ w.r.t. x yields $f'(x) = -((x+1)/x^2)\, \beta^{1/x} \ln \beta + \beta^{1/x} - 1$. As $x \to \infty$, this expression tends to zero. Furthermore, its derivative is positive. Hence $f'(x)$ is negative everywhere and so $f(x)$ is decreasing.

We are now in a position to describe the firm's quality range, and its associated price schedule, in the limit.

Let n be fixed. For the corresponding optimal choice of qualities, we have $r_{k-1,k} = q/(q-1)$. Then, it follows from (2) and (13) that

$$\bar{t}_{k+1} = \bar{p}_1 + \left(\frac{q}{q-1} + k - 1\right)(\bar{p}_2 - \bar{p}_1). \tag{22}$$

Taking a consumer of any given income level $\tau \in [a, b]$, we denote by $k(n)$ the index of the product he buys among the n products available. For $k(n) < n$, we have $\tau \in (\bar{t}_{k(n)}, \bar{t}_{k(n)+1}]$. Now let $n \to \infty$. As $\bar{t}_{k(n)+1} - \bar{t}_{k(n)} = \bar{p}_2 - \bar{p}_1$ tends to zero, we obtain

$$\lim_{n \to \infty} \bar{t}_{k(n)+1} = \tau. \tag{23}$$

Taking the limit of $\bar{p}_{k(n)}$ in (15) for $n \to \infty$ and using (22) and (23), $\lim_{n \to \infty} \bar{p}_{k(n)} = \lim_{n \to \infty} \bar{p}_i + \tau - \lim_{n \to \infty} \bar{p}_1 - \lim_{n \to \infty}((\bar{p}_2 - \bar{p}_1)/(q-1))$. But, by (21), $\lim_{n \to \infty}((\bar{p}_2 - \bar{p}_1)/(q-1)) = au_0/s$, whence

$$\lim_{n \to \infty} \bar{p}_{k(n)} = \tau - \frac{au_0}{s}, \tag{24}$$

which gives the relationship between consumer income and price paid. Notice, in passing, that all consumers have the same income remaining after purchase; but this is a property special to the present example, and does not hold generally.

To complete our characterization, however, we need to calculate the quality of the product consumed by a consumer with income τ. To that end, we take $\lim_{n \to \infty} \bar{u}_{k(n)}$, where $\bar{u}_k = \bar{u}_1 q^{k-1}$. To find the limit of $q^{k(n)-1}$, we consider the limit of $\ln q^{k(n)-1}$. Using (22), (23), and (24) we obtain $\lim[k(n)-1] \ln q = \lim(((\tau - \bar{p}_1)/au_0)\, u_1 - q)(\ln q/(q-1))$. As $n \to \infty$, $(\ln q/(q-1)) \to 1$. Hence, by (3),

$$\lim_{n \to \infty} \bar{u}_{k(n)} = s \exp\left\{\frac{\tau - a}{au_0} \cdot s\right\}. \tag{25}$$

Finally, combining (24) and (25), we get *the optimal price–quality schedule* for the infinite case, viz.,

$$\bar{p}(u) = \frac{au_0}{s} \ln \frac{u}{s} + a\left(1 - \frac{u_0}{s}\right), \tag{26}$$

where s is implicitly defined by (21). (When $u = s$, (26) reduces to (3), where $u_1 = s$.)

This is a *continuous* and *increasing* pricing scheme, and may be illustrated as in Fig. 1. Given this scheme, a consumer with income τ chooses a product of quality u which maximizes his utility $u \cdot [\tau - p(u)]$ on $[s, \bar{u}]$. Consequently, if $\tau < (au_0/s)\ln(\bar{u}/s) + a$ $(= b - au_0/s$ by (21)), the consumer chooses the quality $\tilde{u} = s\exp\{((\tau - a)/au_0)s\} < \bar{u}$; otherwise he buys the top quality product $\bar{u}$.

Therefore, the optimal configuration involves selling the top quality to a band of consumers in the income range $[b - au_0/s, b]$ *and qualities in* $[s, \bar{u})$ *to consumers in the income range* $[a, b - au_0/s)$.

It is worthwhile noticing that when the income range shrinks, the lowest quality s increases monotonically to the top quality $\bar{u}$, and becomes equal to $\bar{u}$ exactly when condition (∗) ceases to be true. Then, condition (∗∗) begins to hold, and only the top quality $\bar{u}$ is sold. This makes the solution for the infinite case consistent with that for the discrete case.

Now the continuous scheme to which we have converged here could obviously be studied directly, by posing the problem: if the firm can choose any interval of products in $[u_0, \bar{u}]$ and set a price schedule $p(u)$, then what interval, and schedule $p(u)$, maximizes profit? It is in fact the case that the limiting scheme we have described here is indeed an optimum in this "continuum" problem, as can be shown using variational methods.

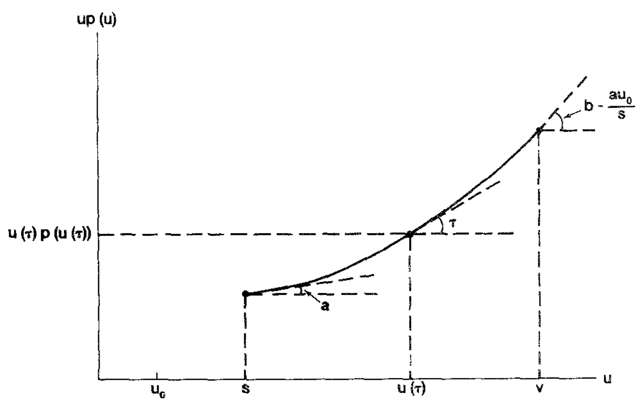

FIG. 1. The relationship between a consumer's income τ, the quality $u(\tau)$ he consumes, and the price $p(u(\tau))$ he pays.

5. Summary and Conclusions

We have investigated the problem faced by a monopolist in choosing a range of product qualities, and an associated price schedule, when selling to a population of consumers who differ in income (willingness to pay). Our central result may be given an intuitive interpretation, as follows. If, as here, it is no more costly to produce a higher quality product, it would seem obvious at first glance that the firm might tend to simply offer a single product, being that of the highest feasible quality, in that consumers' willingness to pay increases with quality. But if the firm does this, then the product will be purchased by all consumers whose income exceeds that of some "marginal consumer" whose reservation price coincides with the price chosen. This, then, means that all richer, intramarginal, consumers enjoy some consumer surplus. But this now suggests another argument: if the firm were to offer more than one quality, charging more for a higher quality, could it not extract some of this consumer surplus?

The central theme of the present paper is that the optimal strategy will be one of two kinds, depending on how broad the range of incomes is. There is a critical income range, such that, if the range of incomes is narrower than this, it is optimal to offer only the top quality product. If, on the other hand, incomes are more widely dispersed, the firm will segment the market by offering the maximum number of qualities permitted.

The intuition which underlies this result is straightforward: for if incomes are "similar," then little consumer surplus is "lost" in offering the top quality product at a price such that even the poorest consumer will buy it. But if incomes are more widely separated, the surplus thereby enjoyed by richer consumers becomes larger—and this makes it more attractive to segment the market.

What is more puzzling is the manner in which this discontinuous shift in strategy occurs at a certain critical range of incomes. We have here resolved this discontinuity by examining what happens to the pattern of market shares (of the various products on offer) in the case where the firm is constrained to sell (at most) n distinct qualities. The associated pricing policy is such that, as the range of incomes narrows towards the critical value, though the firm still continues to offer exactly n distinct qualities, the market share of the top quality product rises continuously as the range of incomes narrows, until it absorbs the entire market, at this critical value.

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